1+1 Dimensional Quantum Field Theory

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In 20th century, two of the greatest discoveries in modern physics are relativity and quantum mechanics. Quantum field theory (QFT), which is built by combining these two theories, is currently the most fundamental description of microscopic world. The first three of four fundamental interactions, electromagnetic, strong, weak, and gravitational interactions, are described by the standard model in QFT. Therefore we must understand QFT in order to deeply understand of physics.

In this lecture, a different approach is used. Instead of usual technical approach in regular field theory class, we demystify QFT by harmonic oscillators. For simplicity, we only consider 1 + 1 dimensions theory. All the machinery can be generalized to 3 + 1 dimensions straightforwardly.
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FIG. 1. One dimensional chain of harmonic oscillators.

I. $N$–COUPLED OSCILLATORS

A. 1D chain (ring)

Consider a one-dimensional chain of $N$-coupled harmonic oscillators. If the chain is long enough, the dynamics of the system should be insensitive to the boundary condition. Therefore, we label oscillators by $n = 1, 2, \ldots, N$, with periodic boundary condition such that $n = 0$ and $N$ are identical. This one dimensional chain is identical to a ring.

Each oscillator has one dimensional coordinate $x_n = na$, where $a$ can be viewed as the basic length unit. The displacement of the oscillator away from its equilibrium point is denoted by $q$. The total kinetic energy is

$$T = \frac{1}{2} m \sum_{n=1}^{N} \dot{q}^2(na) \quad (1)$$

where the dot above $q$ is time derivative. The total potential energy is ($n = N + 1 = 1 \mod N$)

$$V = \frac{1}{2} \kappa \sum_{n=1}^{N} \{q(na) - q([n+1]a)\}^2 \quad (2)$$

where $\kappa$ is the “spring” constant between the adjacent oscillators.

With the total kinetic and potential energy, we can write down the Lagrangian of the system.

B. Equation of Motion (EOM)

The EOMs of the system are coupled linear differential equations

$$m \ddot{q}(na) = -\frac{\partial V}{\partial q(na)} \quad (3)$$

$$= -\kappa \{2q(na) - q([n-1]a) - q([n+1]a)\}. \quad (4)$$

From the right hand side of Eq. (4), we observe that the dynamics of a harmonic oscillator inside the chain depends on the interactions of the two adjacent oscillators from both sides. This
result is expected because without these interaction terms, the system becomes \( N \) independent oscillators, which is trivial.

To solve Eq. (4), which contains \( N \) coupled linear equations, we can diagonalize these equations by introducing the normal coordinates \( u_{kl} \), which is defined through the discrete Fourier transformation

\[
q(na) = \frac{1}{\sqrt{N}} \sum_{kl} e^{ikl na} u_{kl}
\]  

(5)

where

\[
k_l = \frac{2\pi}{Na} l \quad \text{with } l = 0, \pm 1, \pm 2, \ldots, \frac{N}{2}.
\]  

(6)

\( l \) must be integer and \( l = 0 \) corresponds to zero mode. Note that Eq. (5) with Eq. (5) satisfies the periodic boundary condition.

There is always one zero mode \( l = 0 \), which corresponds to all coordinates moving together. It is a free motion and the potential energy is zero. Besides zero mode; for \( N = 3 \), there are two additional modes corresponds to \( l = \pm 1 \); for \( N = 4 \), there are three additional modes corresponds to \( l = \pm 1 \) and 2 (the mode \( l = -2 \) is the same as \( l = 2 \)).

By definition Eq. (5), the normal coordinates with positive and negative \( l \)'s are complex conjugate of each other

\[
u_{kl} = u_{-kl}^*
\]  

(7)

with opposite chirality.

II. NORMAL MODE EXPANSION AND SOLUTIONS

A. Normal mode dynamics

The Lagrangian of the normal modes are

\[
L = \frac{m}{2} \sum_{kl} \dot{u}_{kl} \dot{u}_{-kl} - \frac{\kappa}{2} \sum_{kl} 2[1 - \cos(kla)]u_{kl}u_{-kl}.
\]  

(8)

Introduce the canonical coordinates

\[
p_{kl} = \frac{\partial L}{\partial \dot{u}_{kl}} = m\dot{u}_{kl}
\]  

(9)

\[
p_{-kl} = \frac{\partial L}{\partial \dot{u}_{-kl}} = m\dot{u}_{-kl}
\]  

(10)
FIG. 2. Dispersion relation of one dimensional harmonic oscillator with periodic boundary condition.

and the resulting Hamiltonian is a sum of non-interacting normal modes

\[ H = \sum_{k_l} \left( \frac{1}{2m} p_{k_l} p_{-k_l} + \frac{1}{2} m \omega_{k_l}^2 u_{k_l} u_{-k_l} \right). \] (11)

Amazingly, in the normal mode coordinate, the EOM’s decouple with different normal frequency \( \omega_{k_l} \).

B. Dispersion relation and quantization

Dispersion relation: frequency \( \omega \) related to different \( k \)

\[ \omega_{k_l} = \sqrt{\frac{2\kappa |1 - \cos (k_l a)|}{m}} = 2 \sqrt{\frac{\kappa}{m}} \left| \sin \left( \frac{k_l a}{2} \right) \right|. \] (12)

See Fig. 2 One can introduce the creation and annihilation operators

\[ \hat{a}_{k_l} = \sqrt{\frac{m \omega_{k_l}}{2\hbar}} \left( \hat{u}_{-k_l} + \frac{i}{m \omega_{k_l}} \hat{p}_{k_l} \right) \] (13)

\[ \hat{a}_{k_l}^\dagger = \sqrt{\frac{m \omega_{k_l}}{2\hbar}} \left( \hat{u}_{k_l} - \frac{i}{m \omega_{k_l}} \hat{p}_{-k_l} \right) \] (14)

such that we have \( N \) non-interacting harmonic oscillators

\[ H = \sum_{k_l} \mathcal{H}_{k_l} \] (15)
where

\[ H_{ki} = \hbar \omega_{kl} \left( \hat{a}_{kl}^{\dagger} \hat{a}_{kl} + \frac{1}{2} \right) \]  \hspace{1cm} (16)

It is interesting to note that even though every term of potential energy seems to support an oscillator with angular frequency \( \omega = \sqrt{\kappa/m} \), the normal modes can have a range of angular frequency, going from 0 to \( 2\omega \).

C. Quantum states

The ground state of the system is when all oscillators are the ground state

\[ |0, 0, \ldots, 0\rangle \quad \text{with } E_0 = \frac{\hbar}{2} \sum \omega_{kl} \text{ (vacuum energy)} \]  \hspace{1cm} (17)

and its the wave function is the product of all ground state wave function of the individual harmonic oscillator

\[ \prod_{ki} \phi_0(u_{kl}) \]  \hspace{1cm} (18)

which is a complicated function of the original coordinates \( q \)'s.

The first excited state is a set of states with one quantum in one of the oscillators \( (k_l) \)

\[ |0, \ldots, 1, \ldots, 0\rangle \quad \text{with } E_{k_l} = E_0 + \hbar \omega_{k_l} \]  \hspace{1cm} (19)

which has the excited state energy \( \Delta E_{k_l} = \hbar \omega_{k_l} \). Only the excitation energy is measurable experimentally.

III. CONTINUUM LIMIT

A. Taking continuum limit

Let \( a \rightarrow 0 \) and \( N \rightarrow \infty \), but keep \( Na = L \) finite. We have infinite number of quantum mechanical degrees of freedom (field theory). This is reason why quantum field field theory is so difficult.

We define a field through

\[ q(x,t) = \lim_{a \rightarrow 0} \frac{q_n(t)}{\sqrt{a}} = \lim_{N \rightarrow \infty} \frac{1}{\sqrt{Na}} \sum_k u_k(t)e^{ikx} = \frac{1}{\sqrt{L}} \sum_k u_k(t)e^{ikx} \]  \hspace{1cm} (20)

\[ p(x,t) = \lim_{a \rightarrow 0} \frac{p_n(t)}{\sqrt{a}} = \lim_{N \rightarrow \infty} \frac{1}{\sqrt{Na}} \sum_k p_k(t)e^{-ikx} = \frac{1}{\sqrt{L}} \sum_k p_k(t)e^{-ikx} \]  \hspace{1cm} (21)
In the $a \to 0$ limit, we pack infinite number of degrees of freedom in the finite line segment $L$.

Correspondingly, there are infinite number of non-interacting normal modes

$$k = \frac{2\pi}{L}l \quad \text{with} \quad l = 0, \pm 1, \pm 2, \ldots, \infty.$$  \hspace{1cm} (22)

Now $\omega = \omega_0 k$; $\omega_0 = \sqrt{\kappa/m}$ and $k$ is still discrete. $\omega a$ has a unit of velocity $v_s$, which is the sound speed in this one dimensional medium. Thus

$$\omega = v_s k.$$  \hspace{1cm} (23)

B. Wave equation

The classical EOM in the continuum limit becomes the wave equation

$$\left( \frac{1}{v_s^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) q(x,t) = 0$$  \hspace{1cm} (24)

whereas

$$k = \frac{\omega}{v_s} = \frac{2\pi}{\lambda}$$  \hspace{1cm} (25)

where $\lambda$ is the wavelength.

Thus in this finite-length $L$, one dimensional system (a string), with infinite number of harmonic oscillators, one can equivalently represent the system by infinite number of waves with variable $k$.

$q$ is the wave field. Large $k$ means small wavelength (UV mode) and small $k$ means large wavelength (IR mode); smallest $k$ are $\pm 2\pi/L$ and 0.

IV. 1+1 DIMENSIONAL QUANTUM FIELD THEORY

One dimensional field theory deals with this 1 dimensional systems of waves. In the previous classical example, we have free waves, i.e., the waves do not interact. However, more meaningful examples deals with waves that interact. We can easily add interactions when using Lagrangian dynamics for the field theory.
A. Relativistic waves

When we are dealing with fundamental theories, we know that relativity is important. Therefore, the wave equation must be invariant under Lorentz transformation

\[ x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \]  
\[ t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}. \] 

In relativistic theory, the wave equation is invariant if \( v = c \)

\[ \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \phi(x, t) = \left( \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2} \right) \phi(x', t') = 0 \] 

which describe a massless system.

B. Quantum mechanical wave

In quantum mechanics, particles are described by quantum mechanical waves, just like that the electron is described by electron wave. For non-relativistic particles, they are described by waves satisfying Schrödinger equation, which corresponds to

\[ E = \frac{p^2}{2m}. \] 

where \( m \) is the rest mass.

For relativistic quantum mechanics, particle, it shall satisfy the relativistic wave equation. For a free particle, relativistic wave equation shall be derived from

\[ E^2 = p^2 c^2 + m^2 c^4. \] 

Again, \( m \) is the rest mass.

C. Klein-Gordon equation

For the relativistic energy-momentum relation, one can derive the following wave equation

\[ \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2} \right) \psi = 0 \] 

This is famous Klein-Gordon equation. Comparing to our earlier example, one has an extra mass term

\[ \frac{m^2 c^2}{\hbar^2} \]
which has the Planck constant $h$, indicating it is a quantum wave equation. In the massless limit $m \to 0$, it reduces to wave equation Eq. (24). The Klein-Gordon equation also reduces to the Schrödinger equation in small velocity limit.

The Klein-Gordon equation describes a single quantum mechanical particle. However, when including relativity (field theory), one needs “second quantization” to describe creation and annihilation of particles.

D. Natural unit

In dealing with relativistic problem, it is quite common to use the so-called natural unit.

$$h = c = 1$$

which connect three different units in SI system, mass in kg, time in second, and length in meter.

After choosing natural unit,

$$c = \times 10^{23} \text{ fm/sec} = 1,$$

therefore $1 \text{ s} = 3 \times 10^{23} \text{ fm}$ or $1 \text{ fm} = 3.3 \times 10^{-24} \text{ s}$. Also, we have

$$hc = 1 = 197 \text{ fm MeV}.$$  

Thus $197 \text{ MeV} = 1 \text{ fm}^{-1}$. This means $[E] = [p] = \left[ \frac{1}{7} \right] = \left[ \frac{1}{7} \right] = \text{MeV}$

All physical quantities can have eV, keV, or MeV as dimension, which is also called mass dimension. The mass dimension of the action $S$ is 0 ($[S] = 1$) and Lagrangian has mass dimension 1 ($[L] = 1$).

E. Quantum field theory

In relativistic theories, the mass and energy can convert into each other. Thus particles can disappear into energy, and conversely energy can create particles. The single particle quantum mechanics as described by Klein-Gordon equation is useless. One needs a theory which can create and annihilate particles. For this, one needs to discuss quantized wave systems (coupled harmonic oscillators) or infinite degrees of freedom quantum systems or quantum field theory.
F. Quantization of 1+1 wave system

One needs to quantize $1 + 1$ dimensional wave system, which is in a sense already quantum mechanical (it contains Planck constant). One can quantize by assuming the field $\phi(x, t)$ is an operator and find the conjugate field operator $\pi(x, t)$ and postulate commutation relation among quantum fields.

However, for a numerical approach, the above strategy is of little use. One can again use Feynman’s path integral approach. To do this, we need to start with a Lagrangian.

G. Lagrangian of a field

The Lagrangian is a sum over all modes, thus

$$ L = \int L dx $$

(36)

where the Lagrangian density can be written as

$$ L = \frac{1}{2} \phi_t^2 - \frac{1}{2} \phi_x^2 - \frac{1}{2} m^2 \phi^2 $$

(37)

where $\phi(x, t)$ is a field. One can verify that the Euler-Lagrange equation reproduces Klein-Gordon equation.

When quantized, the first excited state of the system with a set of harmonic oscillator angular frequency

$$ \omega^2 = k^2 + m^2 $$

(38)

describes a particle of mass $m$ and momentum $k$.

H. Introducing interactions

One dimensional free field theory is very simple and not interesting. To make a non-trivial field theory, we can introduce an interaction term

$$ L = -\frac{\lambda}{4!} \phi^4 $$

(39)

with $\lambda > 0$, so that the total energy has a lower bound.

We will try to focus on solving this so-called $\phi^4$ theory in the remaining lecture.
I. Dimensional analysis

In 1 + 1 dimensional theory, the mass dimension of the field is zero, $[\phi] = 0$; the Lagrangian density has mass dimension 2, $[\mathcal{L}] = 2$; the coupling constant $\lambda$ also has mass dimension 2, $[\lambda] = 2$.

It can be shown that the system still supports a free propagating wave as the first excited state of the system, corresponding to a “physical particle” with non-trivial internal structure.

J. Calculations to do

The Klein-Gordon equation with $\phi^4$ interaction is very difficult. There is no analytic solution ever been found. One can use perturbation theory weak weak coupling constant. Or calculate some physical quantities on the lattice, such as energy of the first excited state (one particle state).

One can calculate the physical mass $M$ of the free propagating wave as a function of bare mass $m$ and bare coupling $\lambda$. $M$ also depends on the momentum cutoff $\Lambda$ and lattice spacing $a$. It can be shown that $M$ logarithmically depends $a$

$$M(m_0, \lambda, a) \sim \ln a. \quad (40)$$

Next, one can calculate the dispersion relation of the particle satisfying the relativistic relation i.e.

$$E^2 = k^2 + M^2 \quad (41)$$

which is the relationship between momentum and energy of a relativistic particle.

K. Euclidean time

To make numerical calculation possible, one has to use Euclidean time and needs to consider evolution in imaginary time.

V. MASS AND DISPERSION RELATION

A. Ground state and filtering

We label the exact ground state of 1 + 1 dimensional field theory as

$$|0\rangle \quad (42)$$
A quantum wave with momentum $k = 0$ can be generated by

$$\hat{\phi}_{k=0}(\tau = 0)|0\rangle$$

where $\hat{\phi}_k$ is the Fourier mode of the field $\phi(x)$. $\hat{\phi}_{k=0}(\tau = 0)|0\rangle$ can be expressed into a set of exact eigenstates. To project out the first excited state, we multiply time evolution operator $e^{-HT}$. After long “time” $T$,

$$e^{-HT}\hat{\phi}_{k=0}(\tau = 0)|0\rangle \sim e^{-MT}|k = 0\rangle.$$ \hspace{1cm} (44)

Only the first excited state with $k = 0$ remains.

**B. Two-point correlation function**

Now define the two-point correlation function

$$C_2(M, T) = \langle 0|\phi(x, T)\hat{\phi}_{k=0}(\tau = 0)|0\rangle$$

which reduces to

$$C_2(M, T) \sim c e^{-MT}$$

at large $T$.

Thus by studying the large-$T$ behavior of the two-point correlation function, one can get the physical mass $M$.

**C. Calculating dispersion relation**

To find the dispersion relation, $E(k)$, one can calculate the two-point correlation function

$$C_2(k, T) = \langle 0|\phi(x, T)\hat{\phi}_k(\tau = 0)|0\rangle$$

At large $T$, the first excited state with momentum $k$ dominates, which produces the following exponential

$$C_2(k, T) \sim c e^{-E(k)T}$$

where $c$ is an arbitrary constant. One can get the $E(k)$ by checking the leading large-$T$ behavior.

It is non-trivial that the first excited state of interacting field satisfies dispersion relation just like a free particle. One can only verify this property numerically.
D. Lattice implementation

Two-point function as a functional integral

\[ C_2(k, T) = \int [D\phi(x, \tau)]\phi(x, T) \int dy \phi(y, 0) e^{S_E} \]  \hspace{1cm} (49)

where the Euclidean action \( S_E \) is

\[ S_E = \int dx d\tau \left[ \frac{1}{2} \phi_t^2 + \frac{1}{2} \phi_x^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right] \]  \hspace{1cm} (50)

and again \( \lambda \) is positive with dimension 2.

On the lattice, one has \( \phi_{ij} \) degrees of freedom with \( i, j = 1, \ldots, N \) with periodic boundary condition

\[ \phi_{i+N,j+N} = \phi_{ij} \]  \hspace{1cm} (51)

One generate configuration \( \{ \phi_{ij} \} \) using Monte-Carlo method, then the two-point correlation function is calculated on the lattice through

\[ C_2(k, m, T) = \sum \phi(x, T) \sum_y e^{iky} \phi(y, 0). \]  \hspace{1cm} (52)

E. Lattice calculation

We consider field configurations in a two dimension lattice, with \( N \) points in “time” as well as space directions. There are \( N^2 \) points in total. Assume the lattice spacing is \( a \) in both directions. Thus, the size of the box is \( L = Na \). To simulate the theory well, one needs to have

\[ \frac{1}{L} \ll m, \sqrt{\lambda} \ll \frac{1}{a} \]  \hspace{1cm} (53)

where \( 1/a \) is the UV cutoff and \( 1/L \) is IR cutoff. These conditions are physical requirements. The first condition, \( \frac{1}{L} \ll m \), tell us that the box size is much larger than the Compton wave length, while the second condition, \( \sqrt{\lambda} \ll \frac{1}{a} \), indicates that the particle travels as it is in a continuous spacetime.

F. Actual consideration

For two dimension simulation, a reasonable choice is \( N = 100 \). For example, if we choose \( m = 1 \), \( \lambda = 1 \), \( a = 0.1 \), and \( L = 10 \), we can analyze
1. Finite volume effect: One can do the simulation on the same system, but with \( N = 500, \ L = 50 \) and with the same \( a, m, \) and \( \lambda \).

2. Finite lattice spacing effect: Do the same system with \( a = 0.05 \) and \( N = 200 \), or \( a = 0.02 \) and \( N = 500 \) (keep the same \( L, m, \) and \( \lambda \)).

On can see that mass \( M \) will have \( \ln a \) dependence in Eq. (40), which can also be computed in perturbation theory. However, when taking the continuum limit \( a \to 0 \), \( M \) is divergent (UV catastrophe). This indicates that the renormalization is needed.

The continuum limit exists when all physical observables are expressed in terms of \( M \) and \( \lambda \).

G. Calculating the mass as a function of \( a \)

One can find the mass \( M \) using Eq. (46) in the following procedure:

1. Calculate \( C_2 \) for several different \( T \).

2. Plot \( \ln C_2 \) as a function of \( T \).

3. Find the mass \( M \) using Eq. (46).

4. For several different \( a \), plot the relation between \( M^2 \) and \( \ln a \).

We can find the famous UV divergence in the field theory as we mentioned in the previous section. However, this divergence does not affect the physical observables in terms of physical mass and coupling.

H. Verifying the dispersion relation

One can find the dispersion relation using Eq. (48) in the following procedure:

1. For a given \( k \), calculate \( E(k) \) with the procedure in Sec. V G.

2. Calculate \( E(k) \) for different \( k \).

3. Plot \( E^2(k) \) as a function of \( k^2 \).
VI. SOLITON SOLUTION OF $\phi^4$ THEORY

A. Sine-Gordon equation

Sine-Gordon equation has a soliton solution. One can calculate the mass of a quantum soliton.